

Theorem of Expended Power and Finite Element Formulations: Hamiltonian Mechanics Framework

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Within the Hamiltonian mechanics framework described here, we have developed in this paper a general numerical discretization via the finite element method for continuum-elastodynamics applications directly stemming from the theorem of expended power involving a built-in scalar function: namely, the autonomous Hamiltonian ($\mathcal{H}(p, q): T^*Q \rightarrow \mathbb{R}$). To provide alternative viewpoints and new avenues, a differential formulation is proposed here rather than resorting to classical variational formulations in which, historically, most finite element developments dealing with elastodynamics have been traditionally described. We place particular emphasis upon the notion that the scalar function that represents the autonomous Hamiltonian of the continuous-body dynamical systems can be a crucial mathematical function and physical quantity that is a constant of motion in conservative systems; this is in contrast to vector quantities customarily employed with Newton-based formulations for which physical quantities are more difficult to associate. As such, the scalar representation enables one to readily capitalize on theorems such as Noether's to establish symmetry properties so that proof of satisfaction of the discrete system as that of the continuous system can be readily established, unlike traditional practices emanating from Newton-based representations. The proposed concepts emanating from the theorem of expended power inherently involving the scalar function (namely, the Hamiltonian) naturally embody the weak form in space that can be described by a discrete Hamiltonian differential operator, and integrating over a given time interval yields the weighted residual in time statement. Thereby, a novel yet simple space-discrete Hamiltonian formulation proposed here only needs to employ the discrete Hamiltonian differential operator, which provides new and alternate avenues and directly yields the semidiscrete ordinary differential equations in time that can be readily shown to preserve the same physical attributes as the continuous systems for continuum-dynamical applications. Theoretical formulations are shown for selected structural elements for illustration of the basic concepts.

I. Introduction

IT APPEARS that the inclusion of the variations in physics has been the focus of philosophical controversies and misinterpretations, due to the perception that the variational calculus seems to bring a purpose to the flow of natural events [1]. The differential calculus [Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) independently developed the differential calculus and Newton first applied it to physics] or the variational calculus [Leonhard Paul Euler (1707–1783) and Joseph-Louis Lagrange (1736–1813) developed the variational calculus in the 18th century, often called the calculus of variations [2], and they used it extensively to derive the differential equations specifically for mechanical systems] is concerned with the infinitesimal changes of the physical quantities.

For instance, Hamilton's principle in dynamics [1,3–7], which is one of the variational principles, is well known to be a fundamental axiom in mechanics and science. It claims that extremizing the action as a functional, first defined and referred to as the principal function by Hamilton [8–10], leads to Lagrange's equations of motion as governing equations for N -body systems, as well as its ability to derive both the governing equations and boundary conditions in continuum-elastodynamics that are continuous in space and time. However, it has been well known that Hamilton's principle possesses a logic inconsistency, as pointed out in the literature [11–17],

regarding forcing the other end point in time to be also prescribed. In this regard, we have previously presented via Hamilton's law of varying action a detailed formulation leading to consistent numerical discretization with focus on space discretization [18] that is noteworthy, although it is also a variational approach.

Ludwig Eduard Boltzmann (1844–1906) classified holonomic systems into scleronomic (meaning rigid) and rheonomic (meaning flow) [1]. In the rheonomic systems, the introduction of the variation enables one to eliminate the time partial-derivative term. The direction of the differential in the rheonomic systems is different from that of the variation, due to the presence of the partial derivative of any physical quantity depending on time explicitly, with respect to time. Literature cites certain advantages of the variational calculus over the differential calculus in the rheonomic systems; but this has to be further explored. Any advantages disappear in scleronomic systems. Note that the variation is imaginary and a mathematical abstract quantity. Most engineering and science applications fall into the class of the scleronomic systems [19]. In the scleronomic systems, the differential can replace the variation ([20], page 14). Therefore, for engineering and science applications that are scleronomic, the variational approach is not particularly necessary to obtain governing equations or to find approximate solutions, and alternatives such as the differential formulations can also be employed; this is the main point. In holonomic-scleronomic systems, the Hamiltonian becomes autonomous. In particular, the areas of solid mechanics or elastodynamics or continuum mechanics to which the finite element formulations are usually applied are generally scleronomic. As a consequence, variational formulations are not indispensable to such applications, and differential forms are also ideal for several of these situations, as described in this paper.

Amalie Emmy Noether (1882–1935) recognized the presence of symmetry in physics. The symmetry in physics corresponds to invariance properties of the scalar functions such as the Lagrangian and the Hamiltonian of dynamical systems [21–24]. In view of Noether's theorem, we have the statement that the spatial translational invariance of the Lagrangian in the configuration space gives

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