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Numerical Investigation of Constrained Direct Solutions Using Hamilton's Law

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This paper presents numerical results obtained for a dynamic system that is modeled using Hamilton's law of varying action and for which geometric constraints are enforced explicitly. These results are compared with numerical results for the same system that were obtained from ordinary differential equations, differential-algebraic equations, and a formulation based on Hamilton's law in which the constraints were enforced implicitly. This comparison had two objectives: first, to show that numerical results from the formulation based on Hamilton's law and explicit constraints were equal or superior to results from other formulations; second, to verify that explicit constraints must be enforced at the end of the time step. Both objectives were successfully accomplished.

Nomenclature

a = parabola constant
E_i = total energy at time step t_i
$\mathbf{F}$ = external forces acting on the particle
f_c = constraint force magnitude
G = constraint violation
g = gravitational constant
g_1 = displacement constraint
$\mathcal{H}$ = Hamiltonian
$\mathbf{i}$ = unit vector in the x direction
$\mathbf{j}$ = unit vector in the y direction
$\mathbf{k}$ = unit vector in the z direction
m = particle mass
N = number of time steps
$\mathbf{P}$ = particle momentum
p = momentum in the x direction
q = momentum in the y direction
$\mathcal{T}$ = kinetic energy
t = time
t_1 = time at the beginning of a time step
t_2 = time at the end of a time step
u = particle velocity in the x direction
$\mathcal{V}$ = potential energy
v = particle velocity in the y direction
w = particle velocity in the z direction
x = particle lateral coordinate
y = particle longitudinal coordinate
z = particle vertical coordinate
ΔE = change in total energy
Δt = time-step duration
$\delta(\,)$ = variation of a quantity
λ = Lagrange multiplier
τ = nondimensional time
$\cdot$ = time derivative
$\bar{\ }$ = value at a time-step midpoint
$\wedge$ = value at a time-step endpoint

I. Introduction

OVER approximately the past 30 years, dozens of researchers have investigated the advantages of using Hamilton's law (or Hamilton's law of varying action) to obtain direct solutions for the response of dynamic systems. In this context, direct solutions refer to those solutions obtained without first deriving a set of differential equations. Bailey [1–4] was one of the first to promote the use of Hamilton's law as a means for obtaining direct solutions for discrete and continuous systems. The practicality of the method in obtaining solutions to difficult problems was demonstrated by Hodges [5], who used it to obtain direct solutions to a Sturm–Liouville problem with discontinuous coefficients. Other examples of the applications of the method include the work of Borri et al. [6], Borri and Mantegazza [7], Hodges and Bless [8], Kunz [9], and others.

Baruch and Riff [10] solidified the theoretical legitimacy and the proper formulation of direct solutions using Hamilton's law when they showed that there existed several possible correct formulations. They also investigated the various aspects involved in obtaining accurate numerical results [11,12]. In a majority of the aforementioned investigations and applications, however, the equations obtained from Hamilton's law were derived specifically for a single dynamic system. To develop a general-purpose multibody analysis capability that incorporates direct solution methods, one would prefer that the governing equations for the dynamic system under investigation be built up from libraries of equations for various types of bodies and joints.

Although the cited investigations demonstrate the accuracy of direct solution methods, there are few, if any, numerical investigations that address the accuracy of direct solution methods applied to dynamic systems with explicit constraints. However, much of the knowledge that has been accrued from investigations using time finite elements to obtain numerical solutions for ordinary differential equations [13–16] is applicable to direct solutions. The reason is that direct solution methods in mixed form can easily be shown to yield similar equations to those one would obtain by applying the method of time finite elements to ordinary differential equations in first-order form.

The objectives of this paper are to address the accuracy of direct solutions using Hamilton's law and to investigate means for quantifying the accuracy of those solutions. In [9], for example, the accuracy of the direct solutions obtained for the dynamics of a double pendulum was quantified by comparing those solutions with a truth solution and by measuring the change in total energy. The truth solution consisted of the solution of ordinary differential equations, the accuracy of which had been quantified using the method proposed by Junkins and Lee [17]. Measuring the change in total energy is practical and is a good measure of accuracy for conservative systems. However, comparison with a truth solution is generally inconvenient, because it involves setting up and solving two identical problems.

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