Journal of Thermophysics and Heat Transfer
Vol. 23, No. 4, October–December 2009

Technical Notes

Second-Order Model for Transient Heat Conduction in a Sphere

Messaoud Guellal*
University of Sétif, Sétif, 19000, Algeria
and
Hamou Sadat†
University of Poitiers, 86022 Poitiers, France

DOI: 10.2514/1.44636

I. Introduction

THE use of simple mathematical formulations to solve the heat conduction problems is of great interest for engineering calculations. Though the traditional analytical models are widely used because of their numerous industrial applications [1–4], they sometimes do not provide satisfaction for large values of Biot number. In addition, these models do not predict the temperature inside the solid (at the center, for example).

In the case of spherical geometries, the analytical solution is expressed as an infinite series which necessitates the numerical solution of a characteristic equation for each value of the Biot number [5]. Moreover, this analytical method sometimes requires the calculation of several hundred terms in the series in order to reach the expected accuracy [6].

A perturbation method has been used in [7] to develop first-order models which can be viewed as improved lumped models for the slab, infinite cylinder, and spherical geometries. A second-order model has been examined in [8] for the slab and in [9] for the cylinder. The aim of this paper is to extend the ideas of [7–9] to the case of a sphere subjected to convection to a surrounding fluid. It is shown that this model gives accurate enough results even for an infinite Biot number. The validity of the model is discussed and the region of the sphere where it can be used is determined.

II. Mathematical Formulation

One considers unsteady heat conduction problems in a spherical solid of radius R, initially at a uniform temperature T_i. The solid is placed in a medium at variable temperature $f(t)$ and a constant convection coefficient h is considered at the external surface of the sphere. It is assumed that the thermophysical properties are constant.

By using the following change of variables:

$$\begin{cases} \theta = \frac{T-T_i}{T_\infty - T_i}, & \tau = Fo = \frac{\alpha t}{R^2}, \\ F = \frac{f-T_i}{T_\infty - T_i}, & Bi = \frac{hR}{\lambda}, \end{cases} \qquad x = \frac{r}{R} \tag{1}$$

where T_∞ represents the constant temperature of the fluid, the heat equation and the boundary and initial conditions can be written as

$$\frac{\partial \theta}{\partial \tau} = \frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2 \frac{\partial \theta}{\partial x}\right) \tag{2}$$

$$\begin{cases} \theta(x,0) = 0, & \text{at } \tau = 0 \\ \frac{\partial \theta}{\partial x} = 0, & \text{at } x = 0 \\ -\frac{\partial \theta}{\partial x} = Bi(\theta - F), & \text{at } x = 1 \end{cases} \tag{3}$$

In the previous equations, T is the temperature, τ the time, r the spatial coordinate, and α and λ are the thermal diffusivity and the conductivity of the solid material, respectively.

When the fluid temperature is constant [$F(t) = 1$], the analytical solution of the problem defined by Eqs. (2) and (3) can be calculated by employing the separation of variables techniques [6]. It is expressed in the form of an infinite series:

$$\theta(x,\tau) = \sum_{i=1}^{\infty} \frac{2(\sin n_i - n_i \cos n_i)}{n_i - \sin n_i \cos n_i} \cdot \frac{\sin(n_i x)}{n_i x} \exp(-n_i^2 \tau) \tag{4}$$

where n_i are the roots of the characteristic equation:

$$\tan n_i = \frac{-n_i}{Bi - 1}$$

To develop a low-order model, we introduce as in [7] the perturbation parameter ε (which will be set to 1 later) in the left-hand side of Eq. (2):

$$\varepsilon \frac{\partial \theta}{\partial \tau} = \frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2 \frac{\partial \theta}{\partial x}\right) \tag{5}$$

We now seek the solution in the following form:

$$\theta(x,\tau) = \sum_{n=0}^{\infty} \varepsilon^n \psi_n(x) \cdot F_n(\tau) \tag{6}$$

$\psi_n(x)$ and $F_n(\tau)$ designate the spatial and time functions at the nth order perturbation, respectively.

By introducing expression (6) into Eq. (5), and equating the different terms, one obtains the following recurrence relations:

$$\frac{\partial}{\partial x}\left(x^2 \frac{\mathrm{d}\psi_0}{\mathrm{d}x}\right) = 0 \tag{7a}$$

$$\psi_{n-1} = \frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \frac{\mathrm{d}\psi_n}{\mathrm{d}x}\right) \qquad n > 0 \tag{7b}$$

$$F_n(\tau) = \frac{\mathrm{d}F_{n-1}}{\mathrm{d}\tau} \qquad n > 0 \tag{7c}$$

The initial and boundary conditions (3), lead one to write

$$\begin{cases} \frac{\mathrm{d}\psi_n}{\mathrm{d}x} = 0, & x = 0 \\ -\frac{\mathrm{d}\psi_0}{\mathrm{d}x} = Bi(\psi_0 - 1), & x = 1 \text{ and } n = 0 \\ -\frac{\mathrm{d}\psi_n}{\mathrm{d}x} = Bi\psi_n, & x = 1 \text{ and } n > 0 \end{cases} \tag{8}$$

By truncating the series (6) at the second order, and using Eq. (7c), the approximate temperature is given as

Received 30 March 2009; revision received 15 June 2009; accepted for publication 16 June 2009.

*Assistant Professor, Laboratoire de Génie des Procédés Chimiques, Route Maâbouda.

†Professor, Laboratoire d'Etudes Thermiques, 40 Avenue du recteur Pineau.